Lecture II: Ito's Formula and Its Uses in Statistical Inference

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Outline

I One View of ”Solving” SDEs and a Refresher on Ito’s Formula

II Extracting Information Implied by a Given SDE Even If Pathwise Solution Unavailable

III Using Techniques Above for Statistical Inference

IV Approximate Solutions and Applications e.g. Test if a Levy Process is Appropriate Given Observational Data Coming from a Complex System

References

For Items I - III Draw from:

Protter, P. (2004) *Stochastic Integration and Differential Equations*, Springer-Verlag, Berlin.
Bass, R. (1997) *Diffusions and Elliptic Operators*, Springer, NYC.
Bibby, B. & Sorensen, M. (1995) *Bernoulli* **1**, 17-39.
Romano, J. & Thombs, L. (1996) *JASA* **91**, 590-600.

For Item IV:

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.

Suppose Given ODE:

$$\frac{dX}{dt} = b^o(X,t)$$

Traditional View of "Solving" Would be To Find A Function Such That (e.g. Use Integrating Factor):

$$X(t) = f(t)$$

"Independent Variable/Index"

Suppose Given SDE:

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

One View of "Solving" Would be
To Find A Function Such That:

$$X_t = f(t, B_t)$$

A Process We Know Well

"Independent
Variable/Index"

Or Suppose Given SDE:

$$dY_t = b^o(Y_t, t)dt + \sigma^o(Y_t, t)dX_t$$

Traditional View of "Solving" would be To Find A Function Such That:

$$Y_t = g(t, X_t)$$

A Process We Know and Love

"Independent Variable"

More Concretely Given SDE:

$$dX_t = \alpha X_t dt + \beta X_t dB_t$$

Standard Example: Geometric Brownian Motion

$$X_t = \exp\left((\alpha - \tfrac{1}{2}\beta^2)t + \beta B_t\right)$$

"Independent Variable/Index"

A Process We Know Well

More Concretely Given SDE:

$$dX_t = \alpha X_t dt + \beta X_t dB_t$$

Found by Introducing: $Y(X) = \log(X)$

and Applying Ito's Formula to Change Variables

$$X_t = \exp\left((\alpha - \tfrac{1}{2}\beta^2)t + \beta B_t\right)$$

"Independent Variable/Index"

A Process We Know Well

Ito's Formula

- One of the Most Widely Known Results Associated with SDEs:

$$f(t, X_t) - f(t, X_o) =$$

Something Unique to Stochastic Integration a la Ito

$$\int_0^t \frac{\partial f(s,X_s)}{\partial s} ds + \int_0^t \frac{\partial f(s,X_s)}{\partial X} dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f(s,X_s)}{\partial^2 X} d[X,X]_s$$

For a General Diffusion:

$$d[X,X]_t = \sigma^2(t,X)dt$$

For Brownian Motion:

$$d[B,B]_t = dt$$

Ito's Formula

- One of the Most Widely Known Results Associated with SDEs (For Time Homogeneous Functions):

$$f(X_t) - f(X_o) =$$

Something Unique to Stochastic Integration a la Ito

$$\int_0^t \frac{\partial f(X_s)}{\partial X} dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f(X_s)}{\partial^2 X} d[X,X]_s$$

A More Fundamental Introduction On Quadratic Variation In The Next Lecture

Ito's Formula

- Result Also Applies to Jump Processes:

$$f(X_t) - f(X_o) =$$

$$\int_{0^+}^{t} \frac{\partial f(X_{s-})}{\partial X} dX_s + \frac{1}{2} \int_{0^+}^{t} \frac{\partial^2 f(X_{s-})}{\partial^2 X} d[X,X]_s +$$

$$\sum_{0<s\leq t} f(X_s) - f(X_{s-}) - \frac{\partial f(X_{s-})}{\partial X} \Delta X_s - \frac{1}{2} \frac{\partial^2 f(X_{s-})}{\partial^2 X} (\Delta X_s)^2$$

Another Illustrative Application of Ito’s Formula

$$dX_t = 0dt + 1dB_t$$

$$Y = f(X) = X^2/2$$

Same Result in Terms of Newly Defined Variables

“Brownian Variables/Coordinates”

$$\int_0^t B_s dB_s = \tfrac{1}{2}(B_t^2 - t) \qquad \text{or } Y_t = \int_0^t X_s dX_s + \tfrac{1}{2}t$$

But Explicit Solutions Are Elusive

And Even For Cases That Are “Solvable” It May Not Be As Nice As We Think…..

Complicated Functions of B.M. Not Always Easy to Express in Terms of Normal Density or Distribution (Think Dickey Fuller Tables)

But Explicit Solutions Are Elusive

However….Ito's Formula Can Be Used to Extract Useful Information, e.g. Moments or a Transition Density

(Different Than Solving in Terms of "Paths")

Ito's Formula is Very Useful In Statistical Modeling Because it Does Allow Us to Quantify Some Properties Implied by an Assumed SDE

Cox Ingersoll Ross (CIR) Process

$$dX_t = \kappa(\alpha - X_t)dt + \sigma\sqrt{X_t}dB_t$$

Rewrite Above Using New Constants

$$dX_t = (a + bX_t)dt + \sigma\sqrt{X_t}dB_t$$

Then Integrate from t to T (Assume X_t Known Deterministically)

$$X_T - X_t = \int_t^T (a + bX_s)ds + \int_t^T \sigma\sqrt{X_s}dB_s$$

Bibby, B. & Sorensen, M. (1995) *Bernoulli* **1**, 17-39.

Cox Ingersoll Ross (CIR) Process

$$X_T - X_t = \int_t^T (a + bX_s)ds + \int_t^T \sigma\sqrt{X_s}dB_s$$

Then Take Conditional Expectation

$$\mathbb{E}[X_T - X_t|\mathcal{F}_t] = \mathbb{E}[\int_t^T (a + bX_s)ds + \int_t^T \sigma\sqrt{X_s}dB_s|\mathcal{F}_t]$$

$$\mathbb{E}[X_T - X_t|\mathcal{F}_t] = \mathbb{E}[\int_t^T (a + bX_s)ds|\mathcal{F}_t] \;+\; 0$$

Brownian Increments are Mean Zero Martingales

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Cox Ingersoll Ross (CIR) Process

$$dX_t = (a + bX_t)dt + \sigma\sqrt{X_t}dB_t$$

$$\mathbb{E}[X_T - X_t|\mathcal{F}_t] = \mathbb{E}[\int_t^T (a + bX_s)ds|\mathcal{F}_t] \; + \quad 0$$

Define: $Y(T) = \mathbb{E}[X_T|\mathcal{F}_t]$

Then Differentiate Deterministic Function Above

$$\frac{dY(T)}{dT} = a + bY(T)$$

$$Y(T) = Y(t)\exp(b(T-t)) + \tfrac{a}{b}\big(\exp(b(T-t)) - 1\big)$$

Bibby, B. & Sorensen, M. (1995) *Bernoulli* **1**, 17-39.

Cox Ingersoll Ross (CIR) Process

$$dX_t = (a + bX_t)dt + \sigma\sqrt{X_t}dB_t$$

Conditional Independence of B.M and Ordinary Differential Equation Results (Integrating Factor) Provides Conditional Mean

$$Y(T) = \mathbb{E}[X_T | \mathcal{F}_t]$$

$$\frac{dY(T)}{dT} = a + bY(T)$$

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Cox Ingersoll Ross (CIR) Process

$$dX_t = (a + bX_t)dt + \sigma\sqrt{X_t}dB_t$$

Conditional Independence of B.M and Ordinary Differential Equation Results (Integrating Factor) Provides Conditional Mean

Conditional Variance Can Be Readily Found by Ito Formula and Well-Known Statistical Identity

$$Z(T) = \mathbb{E}[X_T^2|\mathcal{F}_t] \qquad Y(T) = \mathbb{E}[X_T|\mathcal{F}_t]$$

$$VAR[X_T|\mathcal{F}_t] = Z(T) - \big(Y(T)\big)^2$$

Bibby, B. & Sorensen, M. (1995) *Bernoulli* **1**, 17-39.

$$f(X) = X^2 \qquad [X,X]_t = \sigma^2 X_t$$

Cox Ingersoll Ross (CIR) Process

$$dX_t^2 = 2X_t dX_t + \sigma^2 X_t dt$$

$$dX_t^2 = 2(aX_t + bX_t^2)dt + 2\sigma X_t^{3/2} dB_t + \sigma^2 X_t dt$$

Rewrite in Integrated Form, Taking Conditional Expectations and Then Differentiating Again Yields:

$$\frac{dZ(T)}{dT} = (2a + \sigma^2)Y(T) + 2bZ(T)$$

Bibby, B. & Sorensen, M. (1995) *Bernoulli* **1**, 17-39.

Useful for Method of Moments or Martingale Estimating Equations

$$b_t = \kappa \frac{X_t}{\sqrt{1+X_t}}$$

Approach Allows Estimation and Useful Approximations (Can Handle Cases Where MLE Is Problematic [Hyperbollic Diffusion])

Though Asymptotic Variance Maybe Inefficient

Bigger Problem Lies with Assessing Various Model Assumptions via Goodness-of-Fit (Return to This Point Later….)

Bibby, B. & Sorensen, M. (1995) *Bernoulli* **1**, 17-39.

For Likelihood Based Inference One Requires Transition Density

Maximum Likelihood is Useful for Asymptotically Efficient Inference

However Assessing the Validity of "Independent Increments" Assumption Benefits from Using Information Implied by SDE

Transition Density Can Also Be Obtained by Ito's Formula (Using Backward and Forward Kolmogorov Equations)

Ito's Formula and PDEs

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

An Associated Cauchy PDE:

$$\frac{\partial u(x,t)}{dt} = \mathcal{A}[u(x,t)] \quad \forall t > 0$$

$$u(x,0) = f(x)$$

$$\mathcal{A}[f(X)] = b(X)\frac{\partial f(X)}{\partial X} + \sigma^2(X)\frac{1}{2}\frac{\partial^2 f(X)}{\partial^2 X}$$

Can Solve Deterministic Problem By Taking Expectation Over SDE Paths (See Board)

PDEs Help Characterize SDEs

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

An Associated Cauchy PDE:

$$\frac{\partial u(x,t)}{dt} = \mathcal{A}[u(x,t)] \quad \forall t > 0$$

$$u(x,0) = f(x)$$

$$\mathcal{A}[f(X)] = b(X)\frac{\partial f(X)}{\partial X} + \sigma^2(X)\frac{1}{2}\frac{\partial^2 f(X)}{\partial^2 X}$$

If PDE Solvable, Can Compute "Pay-Off" Functions.
Smooth Adjoint Solutions Yield Fokker Planck $p(X_T|X_t)$
Equation [Useful in Likelihood Based Inference].

Though Most PDEs Rarely Admit Closed-Form Solutions

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

An Associated Cauchy PDE:

$$\frac{\partial u(x,t)}{dt} = \mathcal{A}[u(x,t)] \quad \forall t > 0$$

$$u(x,0) = f(x)$$

$$\mathcal{A}[f(X)] = b(X)\frac{\partial f(X)}{\partial X} + \sigma^2(X)\frac{1}{2}\frac{\partial^2 f(X)}{\partial^2 X}$$

However the Generator Above Can Be Used To Help Approximate PDE Solutions [More Later (Maybe)…..]

Testing Validity of An Implicit Levy Process Assumption

Assume Ornstein Uhlenbeck Process Fits Data

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dB_t$$

Data Generating Process One (Discrete Version of Assumed Model)

$$X_{i+1} = FX_i + \sigma^o Z_i$$

Data Generating Process Two (Discrete Misspecified Model)

$$X_{i+1} = FX_i + \sigma^o Z_i Z_{i-1}$$

$$Z_i \text{ iid Normal Random Variables}$$

Testing Validity of An Implicit Levy Process Assumption

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Data Generating Process Two (Discrete Misspecified Model)

$$X_{i+1} = FX_i + \sigma^o Z_i Z_{i-1}$$

"Noise" Increments Uncorrelated but Not Serially Independent

Z_i iid Normal Random Variables

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Romano, J. & Thombs, L. (1996) JASA 91, 590-600.

Testing Validity of An Implicit Levy Process Assumption

Assume Ornstein Uhlenbeck Process Fits Data

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dB_t$$

Data Generating Process Two (Discrete Misspecified Model)

$$X_{i+1} = FX_i + \sigma^o Z_i Z_{i-1}$$

"Noise" Increments Uncorrelated but Not Serially Independent

$$\phi_i := Z_i Z_{i-1}$$

$$\mathbb{E}[\phi_{i+1}^j \phi_i^k]$$

Forcing Both the Square and Circular Peg Into The Circular Hole

$$\theta \equiv (\alpha, \kappa, \sigma)$$

Assume Ornstein Uhlenbeck Process Fits Data

$$\hat{\theta} \equiv \max_{\theta}\ p(X_0;\theta)p(X_1|X_0;\theta)\ldots p(X_N|X_{N-1};\theta)$$

$$\hat{\theta} \equiv \max_{\theta}\ p(X_0;\theta)p(X_1|X_0;\theta)\ldots p(X_N|X_{N-1};\theta)$$

Data Generating Process One (Discrete Version of Assumed Model)

$$X_{i+1} = FX_i + \sigma^o Z_i \qquad \{X_i\}_{i=1}^N$$

Data Generating Process Two (Discrete Misspecified Model)

$$X_{i+1} = FX_i + \sigma^o Z_i Z_{i-1} \qquad \{X_i\}_{i=1}^N$$

Forcing Both the Square and Circular Peg Into The Circular Hole

$$\theta \equiv (\alpha, \kappa, \sigma)$$

Assume Ornstein Uhlenbeck Process Fits Data

$$\hat{\theta} \equiv \max_{\theta} \; p(X_0;\theta)p(X_1|X_0;\theta)\ldots p(X_N|X_{N-1};\theta)$$
$$\hat{\theta} \equiv \max_{\theta} \; p(X_0;\theta)p(X_1|X_0;\theta)\ldots p(X_N|X_{N-1};\theta)$$

$$\{X_i\}_{i=1}^{N}$$

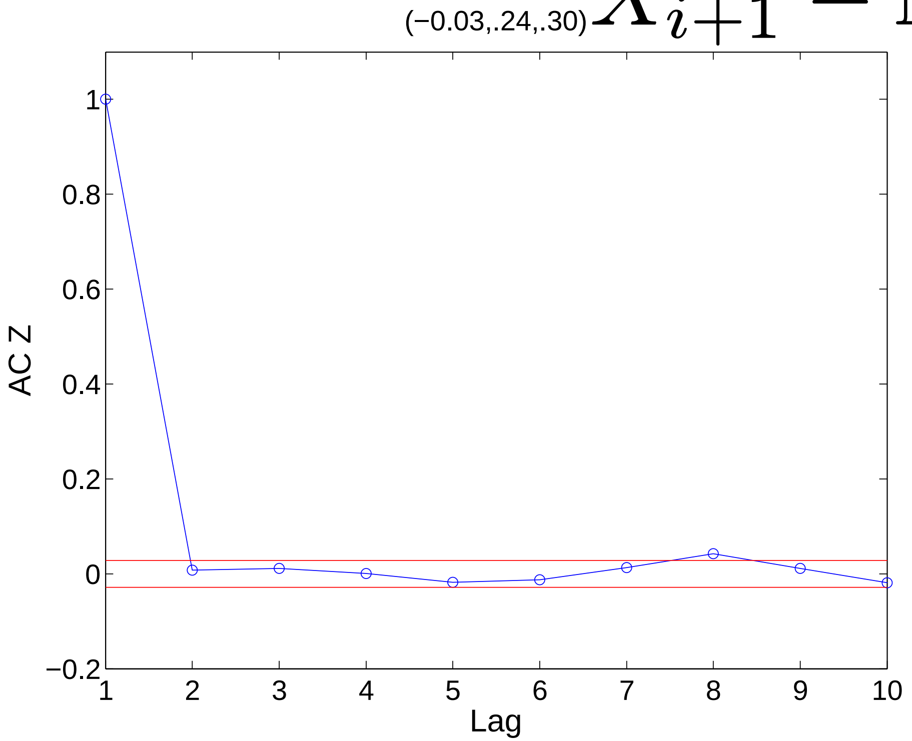
Simulate 100 Time Series Batches Using Common Noise Sequence and Find MLEs

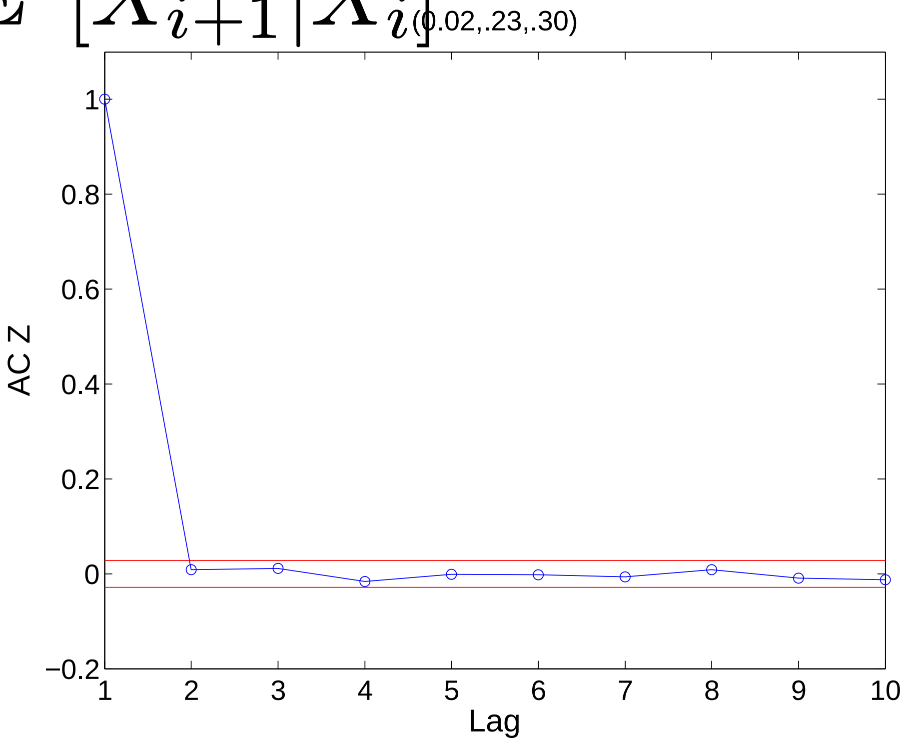
$$\{X_i\}_{i=1}^{N}$$

Forcing Both the Square and Circular Peg Into The Circular Hole

Correlation (A Low Order Moment) In "Residuals" Seems Similar

$$X_{i+1} - \mathbb{E}^{\hat{\theta}}[X_{i+1}|X_i]$$





Correlation (A Low Order Moment) In "Residuals" Seems Similar

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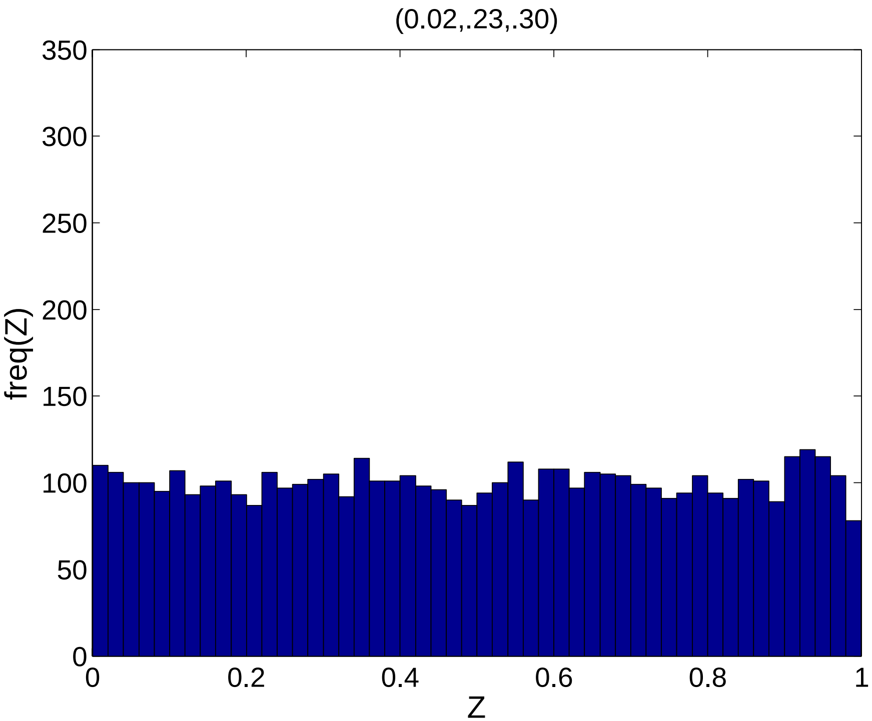

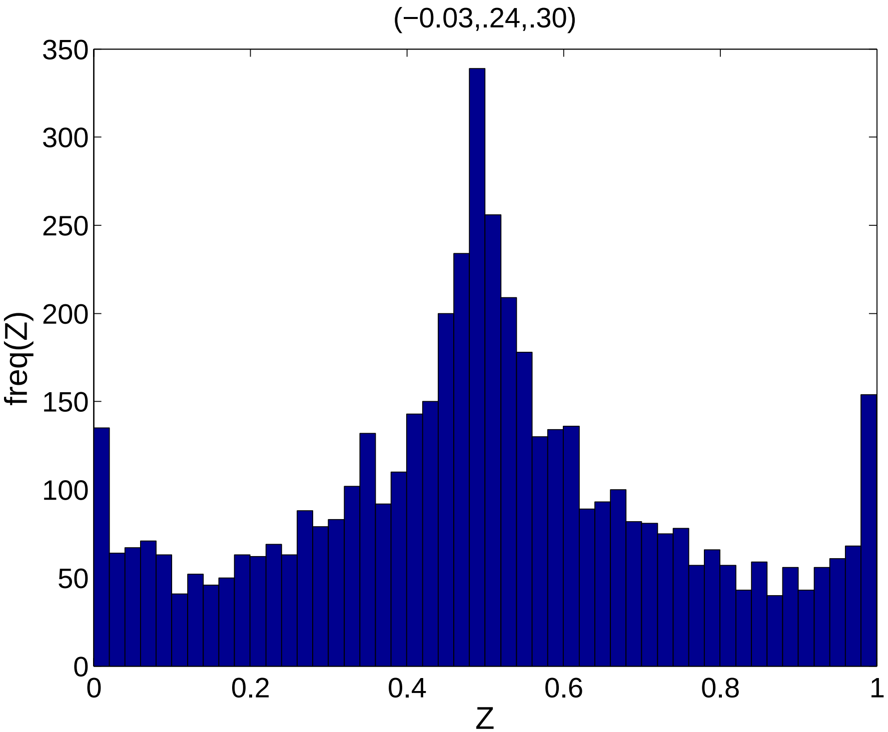

However if Transition Density (All Moments) Used Along with Tools From Lecture 4, the Misspecification Becomes Apparent

The Problem: Transition Density Rarely Available in Closed-form

Many Approaches to Dealing with This Problem (Assess Performance Using Simulated Data of Known Data Generating Process)

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.

The Problem: Transition Density Rarely Available in Closed-form

The Ait-Sahalia Method. Use Generator of Transformed Process to Approximate Transition Density

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

$$Y := \int^{X} \frac{1}{\sigma(u)} du$$

Use Ito to Get a new SDE

(assume well behaved and invertible)

$$dY_t = \mu(Y_t)dt + 1dB_t$$

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.

The Problem: Transition Density Rarely Available in Closed-form

The Ait-Sahalia Method. Use Generator of Tranformed Process to Approximate Transition Density

$$dY_t = \mu(Y_t)dt + 1dB_t$$

$$\mathcal{A}_Y[f(Y)] = \mu(Y)\frac{\partial f(X)}{\partial y} + \frac{1}{2}\frac{\partial^2 f(Y)}{\partial^2 Y}$$

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.

The Problem: Transition Density Rarely Available in Closed-form

The Ait-Sahalia Method. Use Generator of Tranformed Process to Approximate Transition Density

$$\lim_{\delta t \to 0} \delta^{-(J+1)} \left\{ \mathbb{E}[f(Y_{t+\delta})|Y_t = y_o] - \sum_{j=1}^{J} \mathcal{A}_Y^j [f(y_o)] \frac{\delta^j}{j!} \right\}$$

$$= \frac{\mathcal{A}_Y^J [f(y_o)]}{(J+1)!}$$

Various Methods for Exploiting This and Other Related Expansions for Inference

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.

The Problem: Transition Density Rarely Available in Closed-form

The Ait-Sahalia Method. Use Generator of Transformed Process to Approximate Transition Density

$$p_Z(\delta t, z|y_o;\theta) \approx$$

$$\phi(z)\sum_{j=0}^{K}\eta_Z^{(j)}(\delta t, y_o;\theta)H_j(z)$$

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.

The Problem: Transition Density Rarely Available in Closed-form

The Ait-Sahalia Method. Use Generator of Transformed Process to Approximate Transition Density

$$\eta_Z^{(j)}(\delta t, y_o; \theta) \equiv$$
$$\frac{1}{j!}\int_{-\infty}^{\infty} H_j(z) p_Z(\delta t, z | y_o; \theta) dz :=$$
$$\frac{1}{j!}\mathbb{E}[H_j\left(\delta t^{-\frac{1}{2}}(Y_{t+\delta t} - y_o)\right) | Y_t = y_o; \theta].$$

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.

The Problem: Transition Density Rarely Available in Closed-form

Highly Recommended to Test (Both Estimation and Inference Results)All Approximations Using Controlled Data.

Can Give Negative Densities and/or Gradients of Log Likelihood Can Act Funny In Points

Calderon (2007). *SIAM Mult. Mod. & Sim.* **6**.

Prakasa Rao, (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London.